

Truncated Painlevé Expansion with Symbolic Computation for a General Kadomtsev-Petviashvili Equation with Variable Coefficients

Bo Tian^a and Yi-Tian Gao^b

^a Dept. of Computer Sciences, and Inst. for Sci. & Eng. Computations, Lanzhou University, Lanzhou 730000, China

^b Inst. for Sci. & Eng. Computations, Lanzhou University, Lanzhou 730000, China

Z. Naturforsch. **51a**, 175–178 (1996); received November 13, 1995

Able to realistically model various physical situations, the variable-coefficient generalizations of the celebrated Kadomtsev-Petviashvili equation are of current interest in physical and mathematical sciences. In this paper, we make use of both the truncated Painlevé expansion and symbolic computation to obtain an auto-Bäcklund transformation and certain soliton-typed explicit solutions for a general Kadomtsev-Petviashvili equation with variable coefficients.

Key words: Truncated Painlevé expansion, Symbolic computation, General variable-coefficient Kadomtsev-Petviashvili equation, Explicit solutions.

Recently there has been remarkable interest in the investigation of variable-coefficient generalizations of the Kadomtsev-Petviashvili (KP) equation, which provide us with more realistic models in various physical situations. Progress on some of those equations has been made in the study of the solitary wave solution, soliton interactions, complete integrability, Lax pair, etc. See, e.g., [1–6].

The development of symbolic computation helps us, in this paper, to explicitly solve for the general Kadomtsev-Petviashvili equation with variable coefficients

$$\begin{aligned} u_t = & b(t)(6u u_x + u_{xxx}) + k_1(t)(x u_x + 2u) \\ & + s_1(t) u_x + [k_2(t) y + s_2(t)] u_y \\ & + 6b(t) f(t) u + x[f'(t) - 3k_1(t) f(t) \\ & - 12b(t) f^2(t)] + F(t) \\ & + 3b(t) g^2(t) D_x^{-1} u_{yy}, \end{aligned} \quad (1)$$

which is a generalization of the standard KdV and KP equations, the x -KdV and c -KdV equations as well as the general KdV and KP equations [7–13]. The Lax pair, Bäcklund transformation, solitary wave solution, and infinite conservation laws of (1) have been obtained [5].

We will concentrate on the general case of

$$\begin{aligned} b(t) & \neq 0 \quad \text{and} \\ g(t) & = \exp \left\{ \int [2k_1(t) - k_2(t) + 12b(t) f(t)] dt \right\} \neq 0. \end{aligned} \quad (2)$$

To begin with, we make use of the expression for

$$v(x, y, t) = D_x^{-1} u(x, y, t) \Rightarrow u(x, y, t) = v_x(x, y, t), \quad (3)$$

from

$$\begin{aligned} v_{xt} = & b(t)(6v_x v_{xx} + v_{4x}) + k_1(t)(x v_{xx} + 2v_x) \\ & + s_1(t) v_{xx} + [k_2(t) y + s_2(t)] v_{xy} \\ & + 6b(t) f(t) v_x \\ & + x[f'(t) - 3k_1(t) f(t) - 12b(t) f^2(t)] \\ & + F(t) + 3b(t) g^2(t) v_{yy}, \end{aligned} \quad (4)$$

which is equivalent to (1).

It is known that the sufficient condition for a partial differential equation (PDE) to be completely integrable is that it possesses the Painlevé property, i.e., as addressed by Weiss, Tabor and Carnevale [14, 15], that the solutions to the PDE, written as

$$v(x, y, t) = \phi^{-J}(x, y, t) \sum_{l=0}^{\infty} v_l(x, y, t) \phi^l(x, y, t), \quad (5)$$

are single-valued in the neighbourhood of a non-characteristic, movable singularity manifold

$$M = \{(x, y, t) | \phi(x, y, t) = 0\}, \quad (6)$$

where J is a natural number to be determined; $v_l(x, y, t)$ and $\phi(x, y, t)$ are analytic functions with $v_0(x, y, t) \neq 0$.

However, it is not necessary, hereby, to investigate the system's complete integrability and/or Painlevé property. Instead, we aim at deriving certain special solutions by virtue of the truncation of the Painlevé

Reprint requests and correspondence to Prof. Yi-Tian Gao.

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expansion, (5), at the constant level term, i.e.,

$$v(x, y, t) = \phi^{-J}(x, y, t) \sum_{l=0}^J v_l(x, y, t) \phi^l(x, y, t), \quad (7)$$

as well as at obtaining the constraints on the coefficient functions, if any, for the aforementioned solutions to exist.

Then, the leading-order analysis gives that $J = 1$, or

$$v(x, y, t) = v_0(x, y, t) \phi^{-1}(x, y, t) + v_1(x, y, t). \quad (8)$$

When substituting (8) into (4) with *Mathematica*, we make the coefficients of like powers of ϕ to vanish, so as to get the set of Painlevé-Bäcklund (PB) equations as follows:

$$\phi^{-5}: v_0 = 2\phi_x, \quad (9)$$

$$\phi^{-4}: \text{the same as above}, \quad (10)$$

$$\begin{aligned} \phi^{-3}: \Gamma = 3bg^2\phi_y^2 - \phi_t\phi_x + yk_2\phi_y\phi_x \\ + s_2\phi_y\phi_x + xk_1\phi_x^2 + s_1\phi_x^2 \\ + 6b\phi_x^2v_{1,x} - 3b\phi_{xx}^2 + 4b\phi_x\phi_{xxx} = 0, \end{aligned} \quad (11)$$

$$\phi^{-2}: \Gamma_x + \Xi \cdot \phi_x \equiv 0, \quad (12)$$

$$\phi^{-1}: \Xi_x \equiv 0, \quad (13)$$

$$\begin{aligned} \phi^0: v_{1,xt} = b(t)(6v_{1,x}v_{1,xx} + v_{1,4x}) \\ + k_1(t)(xv_{1,xx} + 2v_{1,x}) + s_1(t)v_{1,xx} \\ + [k_2(t)y + s_2(t)]v_{1,xy} + 6b(t)f(t)v_{1,x} \\ + x[f'(t) - 3k_1(t)f(t) - 12b(t)f^2(t)] \\ + F(t) + 3b(t)g^2(t)v_{1,yy} = 0, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \Xi = 3bg^2\phi_{yy} + 6bf\phi_x + k_1\phi_x \\ + 6b\phi_{xx}v_{1,x} - \phi_{xt} + yk_2\phi_{xy} + s_2\phi_{xy} \\ + xk_1\phi_{xx} + s_1\phi_{xx} + b\phi_{4x} = 0. \end{aligned} \quad (15)$$

Integrating (19) gives

$$\begin{aligned} v_{1,x}(x, y, t) &= \Psi(y, t) \\ &\equiv \frac{12b(t)f(t)g^2(t)\omega(t) - g^2(t)k_2(t)\omega(t) + 2\alpha e^{\int k_1(t)dt} f(t)s_2(t) + g^2(t)\omega'(t)}{6\alpha e^{\int k_1(t)dt} b(t)g^2(t)} \cdot y + \frac{f(t)y^2}{b(t)g^2(t)} \\ &\quad \cdot \left\{ 2b(t)f(t) + \frac{k_1(t)}{2} - \frac{A[f(t)]}{6} \right\} - \frac{\alpha^2}{6} e^{2\int k_1(t)dt} - \frac{g^2(t)\omega^2(t)}{2\alpha^2 e^{2\int k_1(t)dt}} - \frac{s_1(t)}{6b(t)} + \frac{\xi'(t) - \omega(t)s_2(t)}{6\alpha e^{\int k_1(t)dt} b(t)}, \end{aligned} \quad (23)$$

$$v_1(x, y, t) = \Psi(y, t) \cdot x + \kappa(y, t), \quad (24)$$

where $\kappa(y, t)$ is a differentiable function, and $A = \frac{d}{dt} \ln$ is a differential operator.

The set of equations (2), (3), (8), (9), (11), (14), and (15) forms an auto-Bäcklund transformation, as long as the set is consistent (or solvable). In the following analysis we devote the effort to certain solvable examples.

The next step is to substitute into (11) and (15) the trial solution

$$\phi(x, y, t) = 1 + \exp\{i[\mathcal{H}(t)x + \mathcal{M}(y, t)]\}, \quad (16)$$

where $\mathcal{H}(t) \neq 0$ and $\mathcal{M}(y, t)$ are complex, differentiable functions. The x -linear form is assumed for the simplification of the future work on (11), and the fact that $\mathcal{H}(t)$ is not a function of y comes from the conclusion of (20), as seen below.

Using (16), we make the coefficients of like powers of x in (11) and (15) to vanish, along with the superficial separation of the real terms from the imaginary ones so as to get a set of equations

$$\mathcal{H}k_1 - \mathcal{H}_t = 0, \quad (17)$$

$$2f\mathcal{H} + g^2\mathcal{M}_{yy} = 0, \quad (18)$$

$$\begin{aligned} -b\mathcal{H}^4 + \mathcal{H}^2s_1 - \mathcal{H}\mathcal{M}_t + y\mathcal{H}k_2\mathcal{M}_y \\ + \mathcal{H}s_2\mathcal{M}_y + 3bg^2\mathcal{M}_y^2 + 6b\mathcal{H}^2v_{1,x} = 0, \end{aligned} \quad (19)$$

$$\mathcal{H}_y = 0 \Rightarrow (16). \quad (20)$$

Aiming at soliton-type solutions, we choose \mathcal{H} and \mathcal{M} to be purely imaginary. Equations (17) and (18) thus, after integration, yield

$$\mathcal{H}(t) = i\alpha e^{\int k_1(t)dt}, \quad (21)$$

$$\mathcal{M}(y, t) = -\frac{f(t)\mathcal{H}(t)}{g^2(t)}y^2 + i\omega(t)y + i\xi(t), \quad (22)$$

where $\alpha \neq 0$ is a real constant, and $\omega(t)$ and $\xi(t)$ are real, differentiable functions.

We substitute (24) into (14) with (2) and *Mathematica* for the simplification, and make the coefficients of like powers of y to vanish, so as to see that $\kappa(y, t)$ is at most a 4th-order polynomial, i.e.,

$$\kappa(y, t) = \sum_{n=1}^4 \beta_n(t) y^n, \quad (25)$$

where $\beta_n(t)$'s are differentiable functions of t , and to see that

$$y^2: \quad \beta_4(t) = \frac{f(t)}{216 b^2(t) g^4(t)} \{54 b(t) f'(t) - 360 b^2(t) f^2(t) - 162 b(t) f(t) k_1(t) - 18 k_1^2(t) \\ - 3 k_1(t) A[b(t)] + 9 k_1(t) A[f(t)] + A[b(t)] A[f(t)] + 3 k_1'(t) - A[f'(t)] A[f(t)]\}, \quad (26)$$

$$y: \quad \beta_3(t) = \frac{f(t) s_2(t)}{54 b^2(t) g^4(t)} \{k_2(t) - 9 k_1(t) - 42 b(t) f(t) - A[b(t)] + 2 A[f(t)] + A[s_2(t)]\} \\ + \frac{\omega(t)}{108 \alpha b^2(t) g^2(t)} e^{-\int k_1(t) dt} \{k_2^2(t) - 72 b^2(t) f^2(t) - 36 b(t) f(t) k_1(t) - 6 b(t) f(t) k_2(t) \\ + 3 k_1(t) k_2(t) + k_2(t) A[b(t)] + 12 b(t) f'(t) - k_2'(t) + 6 b(t) f(t) A[\omega(t)] - 3 k_1(t) A[\omega(t)] \\ - 2 k_2(t) A[\omega(t)] - A[b(t)] A[\omega(t)] + A[\omega(t)] A[\omega'(t)]\}, \quad (27)$$

$$y^0: \quad \beta_2(t) = \frac{s_1(t)}{36 b^2(t) g^2(t)} \{6 b(t) f(t) + 2 k_1(t) + A[b(t)] - A[s_1(t)]\} - \frac{F(t)}{6 b(t) g^2(t)} \\ + \frac{\omega^2(t)}{6 \alpha^2 b(t)} e^{-2 \int k_1(t) dt} \{k_2(t) - 9 b(t) f(t) - A[\omega(t)]\} - \frac{f(t) s_2^2(t)}{18 b^2(t) g^4(t)} + \frac{\alpha^2 f(t)}{6 g^2(t)} e^{2 \int k_1(t) dt} \\ + \frac{e^{-\int k_1(t) dt}}{36 \alpha b^2(t) g^2(t)} \{3 k_1(t) s_2(t) \omega(t) - s_2'(t) \omega(t) - 6 b(t) f(t) \omega(t) s_2(t) + k_2(t) s_2(t) \omega(t) \\ + s_2(t) \omega(t) A[b(t)] - 2 s_2(t) \omega'(t) - 6 b(t) f(t) \xi'(t) - 3 k_1(t) \xi'(t) - \xi'(t) A[b(t)] + \xi''(t)\}. \quad (28)$$

To sum up, the application of the truncated Painlevé expansion and symbolic computation leads to an auto-Bäcklund transformation, as discussed before, and a new family of explicit solutions for (1), a general Kadomtsev-Petviashvili equation with variable coefficients, as follows. Combine all the calculations in this paper and obtain

$$u(x, y, t) = \frac{\alpha^2}{2} e^{2 \int k_1(t) dt} \operatorname{sech}^2 \left[\frac{\alpha x e^{\int k_1(t) dt} - \alpha y^2 f(t) g^{-2}(t) e^{\int k_1(t) dt} + y \omega(t) + \xi(t)}{2} \right] + \Psi(y, t), \quad (29)$$

with its integration as

$$D_x^{-1} u(x, y, t) = v(x, y, t) \\ = \alpha e^{\int k_1(t) dt} \left\{ \tanh \left[\frac{\alpha x e^{\int k_1(t) dt} - \alpha y^2 f(t) g^{-2}(t) e^{\int k_1(t) dt} + y \omega(t) + \xi(t)}{2} \right] - 1 \right\} + \Psi(y, t) \cdot x + \sum_{n=1}^4 \beta_n(t) y^n, \quad (30)$$

where $g(t)$, $\Psi(y, t)$, $\beta_4(t)$, $\beta_3(t)$ and $\beta_2(t)$ are respectively given by (2), (23), (26), (27), and (28). On the other hand, the real constant $\alpha \neq 0$, the real differentiable functions $\omega(t)$ and $\xi(t)$, as well as the differentiable functions $\beta_0(t)$ and $\beta_1(t)$, all remain arbitrary.

The set of solutions is soliton-typed, and thus of potential interest in physical sciences.

Acknowledgements

This work has been supported by the Outstanding Young Faculty Fellowship & the Research Grants for the Scholars Returning from Abroad, State Education

Commission of China; the Youth Sci.-Tech. Foundation & Natural Science Foundation, Gansu Prov., China; and the Youth Science Foundation, Lanzhou Univ., China. The text is written by Y. T. Gao, to whom correspondence should be addressed.

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